```
% sqrt_mat.red

% For a given 2 × 2 matrix A ≥ 0 , we shall prove that
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$$\sqrt{A} = \frac{1}{\sqrt{2\sqrt{ac-b^2}+a+c}} \begin{pmatrix} \sqrt{ac-b^2}+a & b \\ b & \sqrt{ac-b^2}+c \end{pmatrix}$$

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%

on div;

% For a given matrix A = (a b; b c) , we shall find a matrix X = (x y; y z) satisfying A = X^2.

%
mat_a:=mat((a,b),(b,c));
mat_x:=mat((x,y),(y,z));

% The equation A = X^2 is translated into algebraic equations
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(1) $$x^2+y^2=a,\ y(x+z)=b,\ y^2+z^2=c$$

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%

mat_xx:=mat_x*mat_x;
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%

% (1) is equivalent to
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(2) $$x=\sqrt{a-y^2},\ y\left(\sqrt{a-y^2}+\sqrt{c-y^2}\right)=b,\ z=\sqrt{c-y^2}$$

We put

(3) $$f(y)=\sqrt{a-y^2},\ g(y)=y\left(\sqrt{a-y^2}+\sqrt{c-y^2}\right)-b,\ h(y)=\sqrt{c-y^2}$$

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%

func_f:=rhs(first(solve(mat_xx(1,1)=a,x)));
func_h:=rhs(first(solve(mat_xx(2,2)=c,z)));
func_g:=sub({x=func_f,z=func_h},mat_xx(1,2))-b;

% We define a function g_1(w) by
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(4) $$g_1(w)=g(y)\left.\frac{\sqrt{a-y^2}-\sqrt{c-y^2}}{y}\right|_{y=1/w}=-b\sqrt{aw^2-1}+b\sqrt{cw^2-1}+a-c$$

```
%

func_g1:=sub(y=1/w,func_g*(sqrt(a-y^2)-sqrt(c-y^2))/y);

% We put
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(5) $$g_2(v)=g_1(w)\Big|_{w=\sqrt{(v^2+1)/a}}=b\sqrt{\frac{-a+cv^2+c}{a}}-bv+a-c$$

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%
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func_g2:=sub(w=sqrt((v^2+1)/a),func_g1);

% Since
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$$g_2(v)=0$$
$$g_2(v)+bv-a+c=bv-a+c$$
$$(g_2(v)+bv-a+c)^2=(bv-a+c)^2$$
$$\frac{b^2}{a}\left(-a+cv^2+c\right)=(bv-a+c)^2$$

we put

(6) $$g_3(v)=\frac{b^2}{a}\left(-a+cv^2+c\right)-(bv-a+c)^2$$

We can solve the equation $g_3(v)=0$ and get two solutions

$$v_1=\frac{a+\sqrt{ac-b^2}}{b}\ ,\quad v_2=\frac{a-\sqrt{ac-b^2}}{b}$$

Here we note that since

$$g_2(v_2)=\sqrt{-2c\sqrt{ac-b^2}+ac-b^2+c^2}+\sqrt{ac-b^2}-c$$

$g_2(v_2)$ is not always equal to 0. So, we can reject v_2.

```
%

func_g3:=(func_g2+b*v-a+c)^2-(b*v-a+c)^2;
solutions:=solve(func_g3,v);
v1:=rhs(first(solutions));
v2:=rhs(second(solutions));
sub(v=v1,func_g2);
sub(v=v2,func_g2);

% Since
```

$$y=\sqrt{\frac{a}{v^2+1}}$$

we have

$$y_1=\frac{b}{\sqrt{2\sqrt{ac-b^2}+a+c}}$$


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%

y1:=sqrt(a/(v1^2+1));

% Hence, we obtain
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$$x_1=f(y_1)\ ,\quad z_1=h(y_1)$$

this gives

$$X_1=\begin{pmatrix} x_1 & y_1 \\ y_1 & z_1 \end{pmatrix}=\frac{1}{d_1}\begin{pmatrix} \sqrt{ac-b^2}+a & b \\ b & \sqrt{ac-b^2}+c \end{pmatrix}$$

where $d_1=\sqrt{2\sqrt{ac-b^2}+a+c}$. Direct computation shows $X_1^2=A$.

```
%

x1:=sub(y=y1,func_f);
z1:=sub(y=y1,func_h);
mat_x1:=mat((x1,y1),(y1,z1));
d1:=sqrt(2*sqrt(a*c - b**2) + a + c);
d1*mat_x1;
mat_x1*mat_x1;

end;
```